Generalized Integer Partitions, Tilings of Zonotopes and Lattices

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Abstract : In this paper, we study two kinds of combinatorial objects, generalized integer partitions and tilings of two dimensional zonotopes, using dynamical systems and order theory. We show that the sets of partitions ordered with a simple dynamics, have the distributive lattice structure. Likewise, we show that the set of tilings of zonotopes, ordered with a simple and classical dynamics, is the disjoint union of distributive lattices which we describe. We also discuss the special case of linear integer partitions, for which other dynamical systems exist. These results give a better understanding of the behaviour of tilings of zonotopes with flips and dynamical systems involving partitions.

Keywords : Integer partitions, Tilings of Zonotopes, Random tilings, Lattices, Sand Pile Model, Discrete Dynamical Systems.

1 Preliminaries

In this paper, we mainly deal with two kinds of combinatorial objects: integer partitions and tilings. An *integer partition problem* is a set of partially ordered variables, and a *partition* is an affectation of an integer to each of these variables such that the order between these values is compatible with the order over the variables. Tilings are coverings of a given space with tiles from a fixed set. We are concerned here with tilings of two dimensional zonotopes (i.e. $2D$-gons) with lozenges, usually called *random tilings* in theoretical physics. These two apparently different kinds of objects have been brought together in [11, 10]. Some special cases were already known from [9, 3, 2] but the correlations between partitions and tilings are treated in general for the first time in [11]. We begin with a study of the structure of the set of solutions to a partition problem, and then use the obtained results and the correspondence between partitions and tilings to study tilings of zonotopes with flips.

We mainly use two tools: dynamical systems and orders. The use of *dynamical systems* allows an intuitive presentation of the results and makes it easier to understand the relations between the different concerned objects. The use of orders is natural since they appear as structures of the sets we study. An *order relation* is a binary relation over a set, such that for all x, y and z in this set, x is in relation with itself (reflexivity), the fact that x is in relation with y and y is in relation with z implies that x is in relation with z (transitivity), and the fact that x is in relation with y and y with x implies $x = y$ (antisymmetry). The set is then a *partially ordered set* or, for short, a *poset*. Now, in a poset, if any two elements have an *infimum*, i.e. a greatest lower element, and a *supremum*, i.e. a lowest greater element, then it is a *lattice*. The infimum of two elements a and b in a lattice L is denoted by $\inf_L(a, b)$, and their supremum is denoted by $\sup_L(a, b)$. We often write simply $\inf(a, b)$ and $\sup(a, b)$ when the context makes it clear which lattice is concerned. A lattice

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is *distributive* if for all a, b and c: $\sup(\inf(a,b),\inf(a,c)) = \inf(a,\sup(b,c))$ and $\inf(\sup(a,b),\sup(a,c)) = \sup(a,\inf(b,c))$. For more details, see [6].

2 The lattices of integers partitions

2.1 Generalized integer partitions

A *generalized integer partition problem*, or simply a *partition problem* [23, 9], is defined by a Directed Acyclic Graph (DAG) $G = (V, E)$ and a positive integer h. A solution of such a problem, called a *partition*, is a function $a : V \rightarrow \{0, 1, \ldots, h\}$ such that $a(v) \geq a(w)$ for all v and w in V such that there is an edge from v to w in G. The integer $a(v)$ is usually denoted by a_v, and the set of all the solutions of a given partition problem (G, h) is $P(G, h)$. The graph G is called the *base* of the partition problem, and h is the height of the problem. We extend here the usual definition by allowing $h = \infty$, which means that the parts are unbounded. For example, if $G_1 = (V_1, E_1)$ is defined by $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $E_1 = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_5)\}$ then a possible solution to the partition problem $(G_1, 3)$ is $(v_1 \mapsto 3, v_2 \mapsto 3, v_3 \mapsto 2, v_4 \mapsto 1, v_5 \mapsto 0)$.

We can obtain all the solutions of a partition problem $(G = (V, E), h)$ with the following dynamical system. We consider a partition a as a *state* of the system, and a_v as a number of grains stored at $v \in V$. The initial state of the system is the empty one: $a_v = 0$ for all $v \in V$. Then, the following rule is iterated:

A grain can be added at $v \in V$ if the obtained configuration is a partition.

In other words, if a is the state of the system, then the transition $a \longrightarrow b$ is possible if $\forall \nu \neq v$, $b_\nu = a_\nu$, $b_v = a_v + 1$ and b is a partition. The state b is then called a *successor* of a. See for example Figure 1.

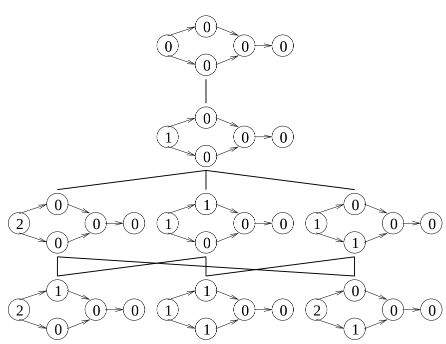


Figure 1: Example of the dynamics with the partition problem $(G_1, 2)$: the first partitions of the set $P(G_1, 2)$ with the possible transitions. We show in each vertex the number of grains it contains. Notice that we do not display all the reachable partitions in this diagram, but only the first ones.

Since there can be no cycle in a sequence of states of the system, this transition rule induces a partial order over the possible states, i.e. over the set of partitions $P(G, h)$. We can now state the main result of this section, which will be useful during the study of tilings problems in Section 3.

Theorem 1 *Given a partition problem $(G = (V, E), h)$, the set $P(G, h)$ equipped with the order induced by the transition rule is a* distributive lattice. *Moreover, the infimum (resp. supremum) of two given partitions a and b in this set is the partition c (resp. d) defined by:*

$$\forall v \in V,\ c_v = max(a_v, b_v),$$

$$\forall v \in V,\ d_v = min(a_v, b_v).$$

Proof : It is clear that c is a partition. Consider now a partition γ. If for one vertex $\nu \in V$, $\gamma_\nu < c_\nu = \max(a_\nu, b_\nu)$ then γ is clearly unreachable from a or b by iteration of the transition rule. Likewise, if for all $\nu \in V$, $\gamma_\nu > c_\nu$ then γ is clearly reachable from c. Therefore, we have that $c = \inf(a, b)$. The proof for $d = \sup(a, b)$ is similar. Therefore, $P(G, h)$ is a lattice. Now, it is easy from these formula to verify that the properties required for a lattice to be distributive are fulfilled.

□

This result says for example that the set $P(G_1, 2)$, partially shown in Figure 1, is a distributive lattice. If $h = \infty$, we obtain an infinite lattice which contains all the possible partitions over the base graph of the problem. Moreover, it is easy to verify that the sets $P(G, h)$ with $h < \infty$ are sub-lattices of the infinite one.

Let us recall that an order *ideal* I is a subset of an order P such that $x \in I$ and $y \leq x$ in P implies $y \in I$. In the finite case, an ideal is defined by a set of uncomparable elements of P and contains all the elements of P lower than any element of this set [6]. From [6], we know that for any distributive lattice there exists a unique order such that the given lattice is isomorphic to the lattice of the ideals of the order, ordered by inclusion. Conversely, the set of ideals such ordered is always a distributive lattice. We will now show that for any partition problem (G, h) with $h < \infty$, $P(G, h)$ is isomorphic to the lattice of ideals of a partial order, which is another way to prove and understand Theorem 1.

First notice that, since $G = (V, E)$ is a Directed Acyclic Graph, it can be viewed as an order over the vertices of G. Now consider the ordered set $\{1, 2, \ldots, h\}$ with the natural order $1 < 2 < \cdots < h$ and the direct product $G \times \{1, 2, \ldots, h\}$ defined by: for all a, b in V, i, j in $\{1, 2, \ldots, h\}$, $(a, i) \leq (b, j)$ if and only if $a \leq b$ in the order induced by G and $i \leq j$. See Figure 2 for an example. Consider now an ideal I of $G \times \{1, 2, \ldots, h\}$. We can define the application p_I, from the set V of vertices of G to $\{0, 1, 2, \ldots, h\}$ by $p_I(v) = j$ where j is the maximal integer i such that $(v, i) \in I$, if any, and $p_I(v) = 0$ if there exists no such i. It is clear that the application defined this way is a partition, and that $r : I \mapsto p_I$ is a bijection between the set of ideals of $G \times \{1, 2, \ldots, h\}$ and the set of partitions $P(G, h)$. Now, we will see that r is actually an order isomorphism by showing that it preserves the covering relation (i.e. the transitive reduction of the order relation) of these orders. Recall that a partition p in $P(G, h)$ is covered exactly by the partitions $p' \in P(G, H)$ such that p' has one more grain at one vertex, say v. This implies that all the vertices v' such that there is a path from v' to v in G must have strictly more grains than v. This means that the corresponding ideals $I = r^{-1}(p)$ and $I' = r^{-1}(p')$ verify $I' \setminus I = \{(v, p_v + 1)\}$, which is exactly the covering relation in the lattice of ideals of $G \times \{1, 2, \ldots, h\}$. Conversely, let us consider two orders ideals I and I' such that $I' \setminus I = \{(v, i)\}$. Then, it is clear that the corresponding partitions $r(I)$ and $r(I')$ only differ by 1 at vertex v.

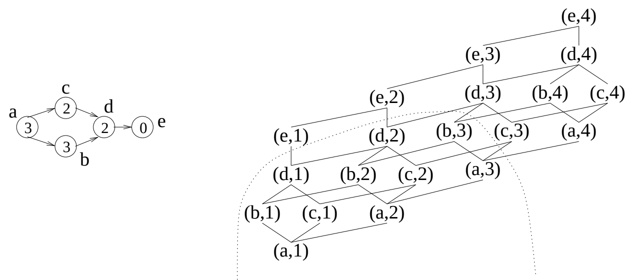


Figure 2: The solutions of (G, h) are nothing but the ideals of $G \times \{1, 2, \ldots, h\}$. Here, we show a graph G (left) with vertices $\{a, b, c, d, e\}$ and the product $G \times \{1, 2, 3, 4\}$ (right), which is equivalent to the partition problem $(G, 4)$. We show the ideal equivalent to the partition displayed on the graph.

We will now use Theorem 1 to study special classes of partition problems, the hypersolid ones, and have a special attention for the so-called linear partitions. However, the reader mostly interested in tilings may directly go to Section 3.

2.2 Hypersolid and Linear partitions

When the base graph of a partition problem is a directed d-dimensional grid then the problem is called *hypersolid* and the solutions are called *hypersolid partitions* [18]. Formally, a d-dimensional grid is a graph $G = (V, E)$ with $V = (\mathbb{N}^*)^d$ such that there is a path from $v = (v_1, v_2, \ldots, v_d)$ to $w = (w_1, w_2, \ldots, w_d)$ if and only if $\forall i, v_i \leq w_i$. For example, the 1-dimensional hypersolid partition problem of length l and height h is $(G = (V, E), h)$ where $V = \{1, 2, \ldots, l\}$ and $E = \{(i, i + 1) | 1 \leq i < l\}$. A 1-dimensional hypersolid partition is called a *linear partition*, and it is denoted by $a = (a_1, a_2, \ldots, a_l)$. A 2-dimensional hypersolid partition is called a *plane partition* and is usually described by an array such that the position (i, j) contains $a_{i,j}$. For example, $\begin{array}{|c|c|c|}\hline 4 & 4 & 2 \\ \hline 3 & 2 & 2 \\ \hline 1 & 1 & 0 \\ \hline\end{array}$ is a solution of the plane partition problem of size 3×3 and of height 5. In this section, we will denote by $H(d, s, h)$ the d-dimensional hypersolid partition problem of size s and height h. Likewise, we denote by $P(H(d, s, h))$ the set of the solutions of $H(d, s, h)$, ordered by the reflexive and transitive closure of the transition relation described in Section 2.1. Therefore, $P(H(1, \infty, \infty))$ is the set of all the linear partitions.

The hypersolid partitions are of particular interest and have been widely studied [1]. Theorem 1 tells us that the sets $P(H(d, s, h))$ are distributive lattices. This is a result which is also known as a consequence of the bijection between $P(H(d, \infty, \infty)$ and the lattice of ideals of $\mathbb{N}^d$ ordered by inclusion [6].

We will now consider the special case of linear partitions. They have been widely studied as a fundamental combinatorial object [1]. As mentioned above, a linear partition of an integer n is simply a decreasing sequence of integers, called *parts*, such that the sum of the parts is exactly n. A linear partition is usually represented by its Ferrer's diagram, a sequence of columns such that if the i-th part is equal to k then the i-th column contains exactly k stacked squares, called grains. In 1973, Brylawski proposed a dynamical system to study these partitions [5]: given a partition a, a grain can fall from column i to column

$i+1$ if $a_i - a_{i+1} \geq 2$ and a grain can slip from column i to column $j > i+1$ if for all $i < k < j$, $a_k = a_i - 1 = a_j + 1$. See Figure 3.

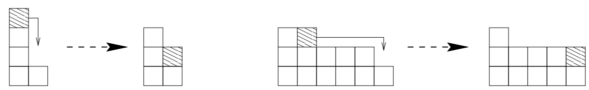


Figure 3: The two evolution rules of the dynamical system defined by Brylawski.

Brylawski showed that the iteration of these rules from the partition (n) gives the lattice of all the linear partitions of n ordered with respect to the dominance ordering defined by:

$$a \geq b \text{ if and only if } \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i \text{ for all } j,$$

i.e. the prefix sums of a are greater than or equal to the prefix sums of b. This lattice is denoted by $L_B(n)$. See Figure 4 (left) for an example. If one iterates only the first rule defined by Brylawski, one obtains the Sand Pile Model and the set of linear partitions obtained from (n) is a lattice, denoted by $SPM(n)$, with respect to the dominance ordering [14]. See Figure 4 (right) for an example. In [16] and [17], it is proved that when these systems are started with one infinite first column the sets of reachable configurations have a structure of infinite lattice, denoted by $SPM(\infty)$ and $L_B(\infty)$. It is also shown in these papers that, if we consider a and b in $SPM(\infty)$ or $L_B(\infty)$ then their infimum c is defined by:

$$c_i = max(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) - \sum_{j > i} c_j \quad \text{for all } i \tag{1}$$

The lattice $L_B(\infty)$ contains all the linear partitions, just as $P(H(1,\infty,\infty))$. We will now study the connection between the dynamical system defined by Brylawski and the one defined in Section 2.1.

Theorem 2 *The application:*

$$\pi_{L_B} : \quad L_B(\infty) \quad \longrightarrow \quad P(H(1,\infty,\infty))$$

such that $\pi_{L_B}(a)_i$ *is equal to* $\sum_{j \geq i} a_j$ *is an order embedding which preserves the infimum.*

Proof : To clarify the notations, let us denote by π the application π_{L_B} in this proof. Let a and b be two elements of $L_B(\infty)$. We must show that $\pi(a)$ and $\pi(b)$ belong to $P(H(1,\infty,\infty))$, that $a \geq_{L_B(\infty)} b$ is equivalent to $\pi(a) \geq_{P(H(1,\infty,\infty))} \pi(b)$ and that $\inf_{P(H(1,\infty,\infty))}(\pi(a),\pi(b)) = \pi(\inf_{L_B(\infty)}(a,b))$. The two first points are easy: $\pi(x)$ is obviously a decreasing sequence of integers for any x, and the order is preserved. Now, let $u = \inf(a,b)$. Then,

$$\begin{aligned} \pi(u)_i &= \textstyle\sum_{j \geq i} u_j \\ &= max(\textstyle\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) \quad \text{from (1)} \\ &= max(\pi(a)_i, \pi(b)_i) \\ &= \inf(\pi(a),\pi(b))_i \end{aligned}$$

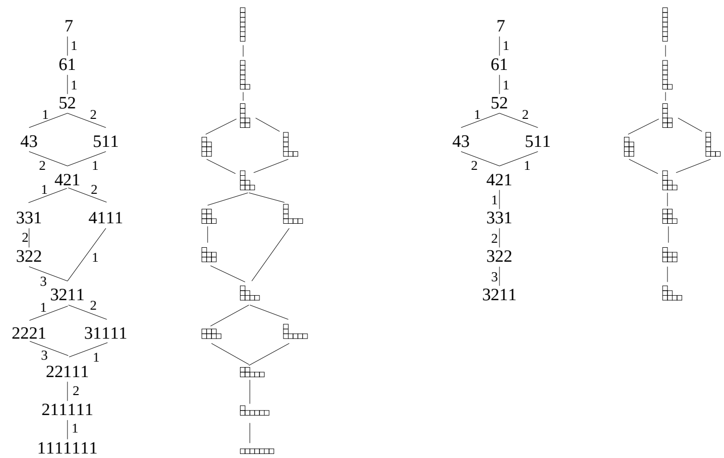


Figure 4: On the left, the diagram of the lattice $L_B(7)$, and on the right the diagram of $SPM(7)$. We showed the representation by piles of grains, and we displayed on each edge the column from which the grain falls during the corresponding transition.

which proves the claim. □

Notice that if we consider the restriction of π_{L_B} to $SPM(\infty)$, denoted by π_{SPM}, a similar proof shows that π_{SPM} is an order embedding which preserves the infimum. However, these orders embeddings are not *lattices* embeddings, since they do not preserve the supremum. For example, if $a = (2,2)$ and $b = (1,1,1)$, then $\pi_{L_B}(a) = (4,2)$, $\pi_{L_B}(b) = (3,2,1)$, $c = \sup_{L_B(\infty)}(a,b) = (2,1)$ but $\pi_{L_B}(c) = (3,1)$ and $\sup_{P(H(1,\infty,\infty))}((4,2),(3,2,1)) = (3,2)$. Notice that there can be no lattice embedding from $L_B(\infty)$ to $P(H(1,\infty,\infty)$ since the fact that $P(H(1,\infty,\infty)$ is a *distributive* lattice would imply that $L_B(\infty)$ would be distributive, which is not true.

3 The lattice of the tilings of a zonotope

A *tiling problem* (see Lecture 7 in [24] for example) is defined by a finite set of tiles T, called the prototiles, and a polygon P. A solution of the problem is a *tiling*: an arrangement of translated copies of prototiles which covers exactly P with no gap and no overlap. The study of these problems is a classical field in mathematics. They appear in computer science with the famous result of Berger [4], who proved the undecidability of the problem of knowing if, given a set of prototiles, the whole plane can be tiled using only copies of prototiles.

We are concerned here with tilings of d-dimensional zonotopes with rhombic tiles. A *d-dimensional zonotope* Z is defined from a family of d-dimensional vectors $\{v_1, v_2, \ldots, v_D\}$ and a set of positive integers $\{l_1, l_2, \ldots, l_D\}$ by:

$$Z = \{\sum_{i=1}^{D} \alpha_i v_i, 0 \leq \alpha_i \leq l_i\}.$$

If $l_i = 1$ for all i, it is also called the *Minkowski* sum of the vectors. Notice that

a two dimensional zonotope is a $2D$-gon (lozenge, hexagon, octogon, etc). The *rhombic tiles* are obtained as the Minkowski sum of d vectors among the ones which generate the zonotope we want to tile. Such a tile is called a *rhombus*. If $d = 2$, they are simply lozenges. Rhombic tilings of zonotopes can be seen as projections of sets of faces of a D-dimensional grid onto the d-dimensional subspace along a generic direction. By construction, the so-obtained tiles are the projections of the d-dimensional facets of the D-dimensional grid and the tiled region is a zonotope. The tiling is then called a $D \to d$ *tiling* and the integer $D - d$ is the *codimension* of the tiling. Figure 5 shows an example of a $3 \to 2$ tiling (left) and a $4 \to 2$ one (right). One can refer to [9, 10, 11, 3, 2] for more details and examples.

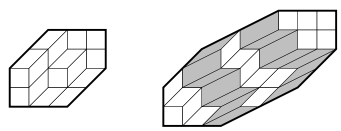
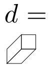
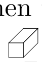
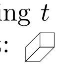
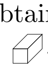


Figure 5: Examples of tilings. Left : a $3 \to 2$ tiling. Right : a $4 \to 2$ tiling. Notice that if we delete the shaded tiles in the $4 \to 2$ tiling then we obtain a $3 \to 2$ tiling.

A dynamical transformation is usually defined over $D \to d$ tilings of a zonotope Z. If we consider the zonotopes obtained as the Minkowski sum of $d + 1$ vectors among the ones which generate Z, then we obtain the most simple sub-zonotopes[2] of Z with non-trivial tilings. Let us call these zonotopes, generated by a family of $d + 1$ vectors, *elementary zonotopes*. One can notice that there are exactly two ways to tile such a zonotope, with $d + 1$ tiles. For example, if $d = 2$ then the elementary zonotopes are hexagons and the possible tilings are  and . Therefore, we can define a dynamics over the tilings of a given zonotope Z: $t \longrightarrow t'$ if and only if we can obtain t' by changing in t the tiling of one elementary sub-zonotope of Z. This means that we locally rearrange $d + 1$ tiles in the tiling t to obtain t'. In the $d = 2$ case, it corresponds to the local rearrangement:  $\longrightarrow$ . If we call t_1 and t_2 the two possible tilings of an elementary zonotope, we will call *flip* the local transformation of t_1 into t_2 in any tiling t, and *inverse* flip the local transformation of t_2 into t_1 in any tiling t. This gives an orientation to the notion of flips, and in the following we will only be concerned with flips (not inverse ones), unless explicitely specified. See Figure 8 for some examples.

In order to continue with the relationship between tilings and partitions, we need to recall the classical notions of de Bruijn surfaces and families. *De Bruijn* grids [7, 8] are dual representations of tilings which have been widely used to obtain important results. De Bruijn grids are composed of the de Bruijn $(d-1)$-dimensional surfaces, which join together the middles of the two opposite sides of each tile. Since the tiles are rhombus, it is always possible to extend these surfaces through the tiling up to the boundary. The set of tiles crossed by a de Bruijn surface is called a *worm*: it is composed of adjacent tiles. If $d = 2$, de Bruijn surfaces are lines, and they join together the middles of the

[2] A sub-zonotope Z' of a zonotope Z generated by $V = \{v_1, \dots, v_D\}$ and $\{l_1, \dots, l_D\}$ is a zonotope generated by a subset $V' = \{v_{i_1}, \dots, v_{i_k}\}$ of V and $\{l'_{i_1}, \dots, l'_{i_k}\}$ with $l'_{i_j} \leq l_{i_j}$ for all j.

opposite edges of the lozenges tiles. An example is given in Figure 6. Each tile is crossed by exactly d de Bruijn surfaces, and there is no intersection of $d+1$ surfaces. On the other hand, there are surfacs which can never intersect, even in an infinite tiling. They join rhombus faces of same orientation, such as lines a and b in Figure 6. We say that these surfaces belong to the same family. A family is equivalent to an edge orientation. In the following, we call *de Bruijn family* the set of tiles in the worms that correspond to the lines of a family. To sum up, we can say that a de Bruijn family of tiles is defined by a vector among the ones which generate the tiled zonotope, and the family contains all the tiles which have this vector as an edge. For example, in Figure 5 (right), a de Bruijn family of tiles is shaded. Notice that deleting such a family in a $D \to d$ tiling gives a $D-1 \to d$ tiling [12]. In the following, when we will consider a $D \to d$ tiling then we will suppose that a family of tiles is (arbitrarily) distinguished, and we will call it the D-th family. We will also denote by $\bar{t}$ the tiling obtained from t when we delete the D-th family of tiles. Therefore, $\bar{t}$ is a $D-1 \to d$ tiling.

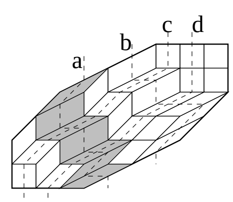


Figure 6: Some de Bruijn lines of a two dimensional tiling. Lines a and b belong to the same family, whereas they are not in the same family as c and d. Moreover, c and d belong to the same family. The shaded tiles represent the worm associated to the line a.

In the following, we will use a variant of the classical de Bruijin lines: the so-called *oriented* de Bruijn lines, which are simply the de Bruijn lines together with an orientation over each line such that for all family, each line in this family have the same orientation. This lead to the usual notion of *dual graph* of a tiling: its set of vertices is the set of intersection points of de Bruijn lines, and there is an edge (i, j) if and only if i and j are in adjacent tiles and there is a piece of de Bruijn line oriented from a to b. Notice that dual graphs of tilings are usually undirected graphs representing the neighbourhood relation of the tiles in the tiling. The two definitions are equivalent, except the orientation of the edges, which is necessary in the following.

Despite our results may be general, we will restrict ourselves to the $d=2$ special case in the following. This means that we tile two dimensional zonotopes (i.e. $2D$-gons) with lozenges. This restriction is due to the fact that these tilings received most of the attention until now, which allows us to use some previously known results which have not yet been established in the general d dimensional case (and which may be false in this case). Notice however that an isomorphism between $d+1 \to d$ tilings, ordered with the transitive and reflexive closure of the flip relation, and d-dimensional hypersolid partitions ordered with the transitive and reflexive closure of the addition of one grain, is exhibited in [9]. Therefore, we can already say from Theorem 1 that the set of $d+1 \to d$ tilings is a distributive lattice.

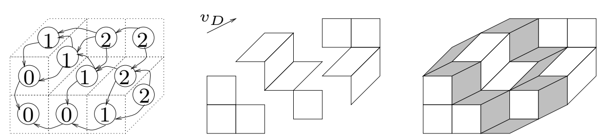


Figure 7: From left to right : A partition which is solution of a partition problem over a graph which is the dual graph of a $3 \to 2$ tiling (dotted), the tiles of this tiling translated with respect to the values of the corresponding parts and the vector v_D, and the obtained $4 \to 2$ tiling after completion. This gives an example of the function $\mathcal{T}$. Notice that we can do the way back from right to left and obtain this way an example for $\mathcal{P}$.

In [11, 9, 10], Destainville studied the relation between rhombic tilings of zonotopes and integer partitions. In the following, we will widely use the correspondence he exhibited. Therefore, we shortly describe it here. For more details, we refer to the original papers.

First, let us see how he associates a partition to a tiling. Let t be a tiling of a zonotope Z. Let us consider $\bar{t}$, the tiling obtained from t by deleting the tiles of the D-th family. Now consider the oriented de Bruijn lines of t. For any tile τ which is not in the D-th family (i.e. a tile in $\bar{t}$), and any de Bruijn line l which is not in the D-th family, we define $w_{\tau,l}$ as the number of de Bruijn lines of the D-th family we cross if we start from the tile τ and follow the de Bruijn line l (with respect to its orientation). Notice that, since the de Bruijn lines of the D-th family define a partition of the tiling into disjoint regions which can not touch two opposite borders of Z, and since the de Bruijn lines go from one border of Z to its opposite, we can always choose the orientations to have $w_{\tau,l_1} = w_{\tau,l_2}$ where l_1 and l_2 are the two de Bruijn lines which cross τ. Therefore, we can denote this value by w_τ. Now, consider the dual graph of $\bar{t}$, $G = (V, E)$. Then, the function p defined for all v in V by $p(v) = w_\tau$, where τ is the dual tile of the vertex v, is a partition solution to the partition problem (G, h) where h is the total number of de Bruijn lines in the D-th family in t. In the following, given a tiling t, we will denote by $\mathcal{P}(t)$ the partition associated this way to t.

Conversely, given a partition p solution of the partition problem (G, h) where G is the dual graph of a tiling t, we want to define a tiling t' associated to p. Let Z be the zonotope tiled by t. Let Z' be the zonotope generated by the same family of vectors than Z with an additional one: v_D with $l_D = h$. Let us consider the following partition of the set of vertices of G (and dually of the tiles of t) : $V_i = \{v \in V \text{ such that } p_v = i\}$. We can now construct t' by insertion of a worm of the D-th family in t between the tiles corresponding to the sets V_i and V_{i+1} for all i. In other words, starting from t, for all i we apply the translation defined by the vector $i \cdot v_D$ to the tiles in V_i, and we add the tiles of the D-th family in order to fill Z'. We obtain this way t', the tiling of Z' associated to p. In the following, given a partition p, we will denote by $\mathcal{T}(p)$ the partition associated this way to p. See Figure 7 for an example.

Destainville obtains this way a method to generate all the $D \to 2$ tilings of a given zonotope Z. He starts from a $3 \to 2$ tiling, which is nothing but the projection of the Ferrer's diagram of a planar partition [9]. Indeed, a planar partition can be viewed as a decreasing sequence of stacked cubes on a 2-dimensional grid, the projection of which gives the $3 \to 2$ tiling. See Figure 5

(left) for an example. From the oriented dual graph of this tiling, he defines a partition problem, the solutions of which are equivalent to $4 \to 2$ tilings, as explained above. Likewise, he can construct a $D+1 \to 2$ tiling from a $D \to 2$ one for any D, and so obtains a way to generate $D \to 2$ tilings for any D. It is shown in [11, 10] that the application $\mathcal{T}$, which generate the tiling associated to a partition is a bijection from the set of the partitions solutions to the problems (G_t, h) for all oriented dual graph G_t of a $D \to d$ tiling to the set of $D+1 \to d$ tilings. Moreover, it is shown in these papers that this bijection is an order isomorphism if the set of partitions is ordered with the reflexive and transitive closure of the relation induced by the additions of one grain (see Section 2), and if the set of tilings is ordered with the reflexive and transitive closure of the relation induced by the flips.

It was shown in [15] and [12] that we can obtain all the two dimensional tilings of a zonotope from a given one by iterating the following rule: the transition $t \to t'$ is possible if the tiling t' can be obtained from t with a flip[3]. This transition rule leads us to consider the tilings of a zonotope as the possible states of a dynamical system. A sequence of such transitions is denoted by $\xrightarrow{*}$, which is equivalent to the transitive and reflexive closure of $\to$, also denoted by $\geq$, depending on the emphasis given to the dynamical aspect or to the order theoretical approach. We denote by $T(Z, D, 2)$ the set of $D \to 2$ tilings of the zonotope Z ordered by $\geq$. An example is given in Figure 8. Notice that all sequences of flips (with no inverse flips) from a tiling to another one have the same length [10], which will be useful in the following. Using the preliminaries given above, we can now state the first results we need to prove that $T(Z, D, 2)$ is a lattice.

Lemma 1 *The set $T(Z, D, 2)$ is the disjoint union of distributive lattices L_i such that a flip transforms a tiling in L_i into another one in L_i if and only if it involves at least one tile of the D-th family. Moreover, for all $t \in L_i$ and $u \in L_j$, $\bar{t} = \bar{u} \Leftrightarrow i = j$.*

Proof : Let us consider the maximal subsets L_i of $T(Z, D, 2)$ such that a flip goes from a tiling in L_i to another one in L_i if and only if it involves at least one tile of the D-th family. It is shown in [11] and [10] that such a set, equipped with the transition rule described above (flip), is isomorphic to the set of the solutions of a partition problem, depending on Z and D, equipped with the transition rule described in Section 2 (addition of one grain). We know from Theorem 1 that this set is a distributive lattice. Therefore, we obtain the first part of the claim.

It is then clear that if s and t are in L_i then $\bar{s} = \bar{t}$: it suffices to notice that if $t \longrightarrow t'$ such that this flip involves at least one tile in the D-th family then $\bar{t} = \bar{t'}$. Moreover if $\bar{s} = \bar{t}$ then s can not be obtained from t with a flip involving three tiles with none of them belonging to the D-th family: such a flip changes the position of the tiles in $\bar{s}$ and $\bar{t}$. This ends the proof. $\square$

Lemma 2 *Let a, b and c be in a L_i (L_i being one of the sets partitioning $T(Z, D, 2)$ defined in Lemma 1) such that a is the unique maximal element of L_i and b is its unique minimal element. If a flip involving three tiles none of*

[3] The generalisation of this claim to any dimension, however, has been pointed out by Reiner as a difficult open question [20].

them belonging to the D-th family is possible from c then it is possible from a and b.

Proof : First notice that, since a flip inside L_i involves tiles which are in the D-th family, we have $\bar{a} = \bar{b} = \bar{c}$. Therefore, the flip from c is possible from a and b if the three tiles it involves are neighbours in a and b. From Lemma 1, $\mathcal{P}(a)$ and $\mathcal{P}(b)$, the partitions which correspond to the tilings a and b, are respectively the maximal and minimal elements of the set $P(G, h)$ of solutions to a partition problem (G, h). Therefore, from Theorem 1, $\mathcal{P}(a)$ and $\mathcal{P}(b)$ have all their parts equal to respectively 0 and h. Then, from the definition of $\mathcal{T} = \mathcal{P}^{-1}$, all the tiles which do not belong to the D-th family tile a sub-zonotope of Z, and so all the flips involving three such tiles are possible from a and b. $\square$

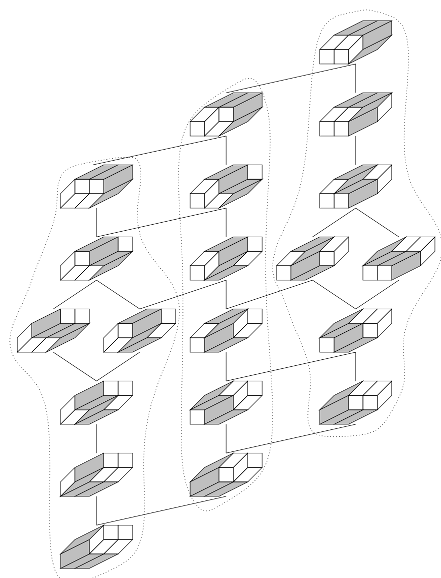


Figure 8: $T(Z, 4, 2)$ for a given Z. The possible transitions (flips) are represented. The shaded tiles show the 4-th family, and the dotted sets are the distributive lattices L_1, L_2 and L_3, as stated by Lemma 1.

We can now define the order $\overline{T(Z, D, 2)}$ as the quotient of $T(Z, D, 2)$ with respect to the equivalence relation $s \sim t \Leftrightarrow \bar{s} = \bar{t}$, i.e. defined by the lattices L_i. In other words, we consider the set of the lattices L_i as the set of vertices of $\overline{T(Z, D, 2)}$, and there is one edge from L_i to L_j in $\overline{T(Z, D, 2)}$ if and only if there is at least a transition from one element of L_i to one element of L_j in $T(Z, D, 2)$. Also notice that if, given a tiling of a zonotope Z, we delete the tiles in the D-th family, we obtain a new zonotope. Notice that this zonotope only depends on Z and does not depend of the considered tiling, since, as one can easily verify, if $t \longrightarrow t'$ then $\bar{t}$ and $\bar{t'}$ tile the same zonotope. Let us denote by Z' this zonotope.

Lemma 3 *The order $\overline{T(Z, D, 2)}$ is isomorphic to the order $T(Z', D-1, 2)$.*

Proof : From Lemma 1, we can associate to each $L_i \in \overline{T(Z, D, 2)}$ a tiling t_i such that for all a in L_i, $\bar{a} = t_i$. It is clear that t_i is in $T(Z', D-1, 2)$. Conversely, if

we have a tiling of Z', then we can use the construction of a $D+1 \to d$ tiling from a $D \to d$ one described above to obtain a tiling t of Z. Therefore, there is a bijection between $\overline{T(Z,D,2)}$ and $T(Z',D-1,2)$. We will now see that it is an order isomorphism. From Lemma 1, if there exists a flip $a \to b$ between two tilings $a \in L_i$ and $b \in L_j$ with $i \neq j$, then it does not involve any tile of the D-th family, and so there exists a flip $t_i \to t_j$ in $T(Z',D-1,2)$. Conversly, if there is a flip $t_i \to t_j$ in $T(Z',D-1,2)$, then there exists $a \in L_i$ and $b \in L_j$ such that $a \to b$: from Lemma 2, it suffices for example to take the maximal elements of L_i and L_j respectively for a and b. Therefore, $\overline{T(Z,D,2)}$ is isomorphic to $T(Z',D-1,2)$. □

With these two lemmas, we have much information about any set $T(Z,D,2)$: it is the disjoint union of distributive lattices, and its quotient with respect to these lattices has itself the same structure, since it is isomorphic to $T(Z',D-1,2)$. This shows that the sets $T(Z,D,2)$ are strongly structured, and makes it possible to write efficient algorithms based on this structure, for example the computation of a shortest sequence of flips which transforms a given tiling into another given one.

4 Conclusion and perspectives

In conclusion, we gave structural results on generalized integer partitions and two dimensional tilings of zonotopes. Our main tools were dynamical systems and order theories. This allows an intuitive presentation of the topic, and the presence of lattices in this kind of dynamical systems seems very general [19]. This makes it possible to develop efficient algorithms for a variety of questions over tilings. We wrote for example an algorithm which transforms a tiling into another one with a minimal number of flips (we do not give it here because of the lack of space). The lattice structure is also strongly related to enumeration problems, and often offers the possibility of giving new combinatorial results. For example, one could use the results presented here to look for a formula for the minimum number of flips necessary to transform a tiling into another one.

There are two immediate directions in which it seems promising to extend the results presented here. The first and obvious one is to study d dimensional tilings with $d > 2$. Then, it is not clear wether all the tilings of a given zonotope can be obtained from a particular one by flipping tiles [20]. Moreover, we would need an efficient formalism in order to give clear and nice proofs. This formalism still has to be developed. The other important remark is that the choice of the D-th family all along our work is arbitrary. This means that we could choose any family to be the D-th, and so there are many ways to decompose $T(Z,D,2)$ into a disjoint union of distributive lattices. This is a strong and surprinsing fact, which has to be fully explored.

Notice also that Remila [21, 22] showed that special classes of tilings with flips are lattices: the domino tilings, the bar tilings and the calisson tilings. Notice that this last class is nothing but the $3 \to 2$ tilings. However, the techniques he used to prove these results are very different from the one presented here, and it would be very interesting to try and extend these results to larger classes of tilings. The idea of considering tilings as projections of some high-dimensional structures seems promising, since it allowed us to study the large class of tilings. Moreover, the bars and dominoes tilings studied by Remila can also be viewed

as projections of high-dimensional objects. Therefore, this approach would be an interesting research direction for general results.

Finally, one may wonder if the results presented here always stands when the support of the tiling is not a zonotope. It would be interesting to know the limits of our structural results. They may be very general, and lead to sub-lattices properties of the obtained sets of tilings. Likewise, it would be useful to study what happens when the size of the zonotope grows to infinity. Some results about that are presented in [11] and [10] but a lot of work remains to be done.

5 Erratum

In a previous version of this paper, we claimed that the sets $T(Z, D, 2)$ were themselves lattices, which is actually false. Indeed, the set $T(Z, D, 2)$ when Z is a $2d$-gon having $l_i = 1$ for all i is isomorphic to the higher bruhat order $B(n, 2)$ (see [13]), and it is known that $B(6, 2)$ is *not* a lattice ([25], Theorem 4.4). We apologize for this, and we thank V. Reiner, who first pointed out this error and gave us useful references.

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